Statics and Dynamics of Anchoring Cables in Waves

Theodore R. Goodman*
Oceanics, Inc., Plainview, N.Y.
and
John P. Breslin†
Stevens Institute of Technology, Hoboken, N.J.

An analysis is presented of the statics and dynamics of mooring cables. The analysis includes effects that have not been considered by any previous investigator and corrects certain errors that have been perpetuated by all previous investigators. In addition to the discussion of hitherto unsuspected physical effects, this paper also includes a discussion of the numerical computation of buoy-cable systems in waves.

Introduction

PRIOR to the publication of Ref. 1, it appears that all studies of cables suspended in a heavy liquid contained an erroneous accounting of the effect of fluid static pressure. It was stated without demonstration in Ref. 1 that the equations generally used by others require replacing of the tension by an apparent tension $T_a = T + \rho g A(h-z)$. It is clear that the apparent tension and the actual tension are equal only at the liquid surface $z = h$ and the tension can vanish and become negative between some point z_l and the bottom of the cable. Engineers have questioned how a cable can sustain negative tensions (axial compressive forces) since they have little (ideally no) bending rigidity.

It is the purpose of this paper to provide the details of the derivation alluded to in Ref. 1. At the same time, a proper accounting will be taken of the effects of extensibility of the cable, and it will be shown that the resulting equations are different in this respect also from the result of all previous investigators. In particular, the apparent tension must itself be replaced by an effective tension $T_e = T + \rho g A(h-z)/(1+\epsilon)$. Moreover, the constitutive equation, which states that the elongation is proportional to the tension, must be replaced by one that states that the elongation is proportional to the effective tension. Thus, the hydrostatic pressure contributes to the elongation by squeezing the cable, and this contribution is sufficient to keep the elongation positive even though the tension itself may become negative. Indeed, a strain gauge attached to the cable will always register a positive value and is only capable of detecting the effective tension, not the actual tension.

Analysis of Static Cable Forces

Forces on an Element

Let us first consider an extensible cable of weight per unit length *w in vacuo* suspended in a heavy liquid of mass density ρ. The cable is considered to be acted upon by a horizontal offsetting force X at the water surface and a vertical force Z at the same point to sustain it in equilibrium; reaction forces X_a and Z_a act at the anchor. A current flowing horizontally and having an arbitrary velocity profile $V_c(z)$ is assumed to exist.

Received May 12, 1975; revision received June 24, 1976. This paper is the result of research sponsored by the NOAA Data Buoy Office under Contract NAS 13-3, although none of the opinions advanced represent the official position of the NOAA Data Buoy Office. The authors would like to thank D.M. Brooks for reviewing the manuscript.

Index category: Marine Mooring Systems and Cable Mechanics.

*Vice President. Associate Fellow AIAA.

†Director, Davidson Lab. Member AIAA.

Figure 1 serves to define the coordinates. Let us now consider the differential element at E, the details of which are exhibited in Fig. 2. Here it is seen that the element of radius a is curved and has a radius of curvature R; the center of curvature lies in the vertical plane. Thus, we restrict the study to cables lying entirely in this plane.

The vertical position of any point on the surface of the cable is

$$z_s = z + a\cos\phi\cos\theta \tag{1}$$

where z is the height of the centerline, a is the cable radius, and θ is the angular coordinate measured in a plane perpendicular to the vertical plane (plane of the paper) from the normal to the intersection of these two planes.

The static fluid pressure at this point on the cable surface is

$$p_s = \rho g(h - z_s) \tag{2}$$

and the pressure can only produce a normal component of force on the cable, the tangential component being zero. If the arc length of the stretched cable element is $(1+\epsilon)ds$, with ds the arc length of the unstretched cable element, and ϵ the extension per unit length at that arc length location s, we can see that the element of area at any angular slice θ is $(R - a\cos\theta)$ $d\phi \cdot a d\theta (1+\epsilon)$ so that the lengths of filament at $\theta = 0$ and $\theta = \pi$ ("top" and "bottom") differ by $2a$. In other words, the fluid pressures have a larger area on which to act on the "lower" side than on the "upper" side. This is basically why the finite radius of curvature produces a nonvanishing contribution to the static normal force density. We do not have to rely on this physical interpretation, but rather proceed to find the differential force density from

$$\frac{dF_n}{d\phi} = -\int_0^{2\pi} p_s \cos\theta (R - a\cos\theta)\, a d\theta (1+\epsilon) \tag{3}$$

in which we make use of Eqs. (2) and (1) to yield

$$\frac{dF_n}{d\phi} = -\rho g a \int_0^{2\pi} [h - z - a\cos\phi\cos\theta](R - a\cos\theta)\ \cos\theta d\theta (1+\epsilon) \tag{4}$$

As only those terms involving $\cos^2\theta$ contribute (producing a factor of π) and all others integrate to zero, we find that

$$\frac{dF_n}{Rd\phi} = \rho g A [\cos\phi + \frac{(h-z)}{R}] = \frac{dF_n}{(1+\epsilon)ds} \tag{5}$$

taking $Rd\phi = (1+\epsilon)ds$, the stretched centerline arc length, and $A = \pi a^2$, the cable cross-sectional area when the cable is stretched.

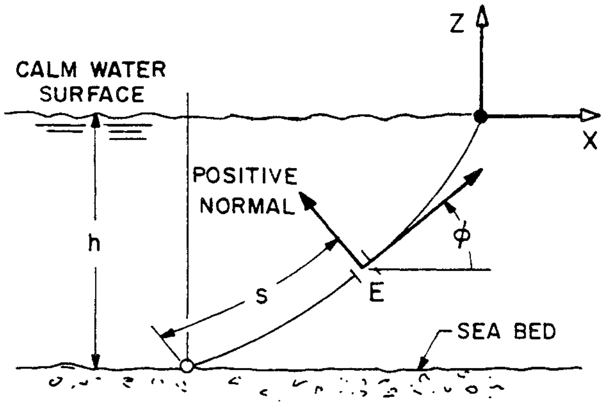

Fig. 1 Global view of cable in static equilibrium.

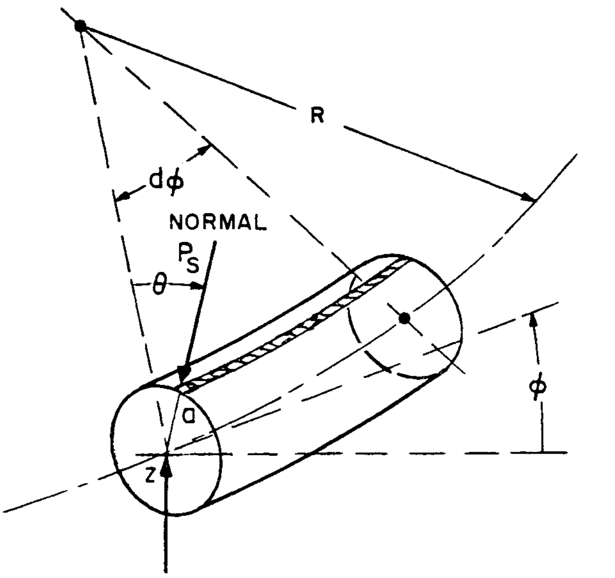

Fig. 2 Geometric definition of curved cable element with centerline at z units above the sea bottom in a fluid of depth h.

It is clear from Eq. (5) that the effect of finite radius of curvature is significant in the normal force arising from liquid pressure at all locations such that $(h-z)/R=0$ (1.0). This is not a sole criterion, since the importance of the normal force must be reckoned in relation to the cable tension which will subsequently be considered.

Before proceeding, it is necessary to note that A is the cable area under load and, hence, is smaller than the unloaded area A_0. The relation between A and A_0 can be determined by considerations in the theory of elasticity.[2] The general relationships between stress and strain for a homogeneous, isotropic material obeying Hooke's law are given in Ref. 2 (p. 8). In cylindrical coordinates r, z, θ, these are

$$\epsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] \tag{6a}$$

$$\epsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_z + \sigma_\theta)\right] \tag{6b}$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] \tag{6c}$$

where ν is Poisson's ratio and E is Young's modulus. The extension per unit length ϵ is actually the longitudinal strain component ϵ_z. Now, consider an element of the cable whose cross section is a circle of radius a and whose length is ds. The stress along the cable element is T/A, whereas the radial stress at the surface $r=a$ caused by hydrostatic pressure is $-\rho g(h-z)$. Reference 2 (p. 216) shows that a solution of the equations of elasticity satisfying these conditions is that the normal stresses are constant throughout the element and the shearing stress is zero. The solution is, therefore

$$\sigma_r = \sigma_\theta = -\rho g(h-z)$$

$$\sigma_z = T/A \tag{7}$$

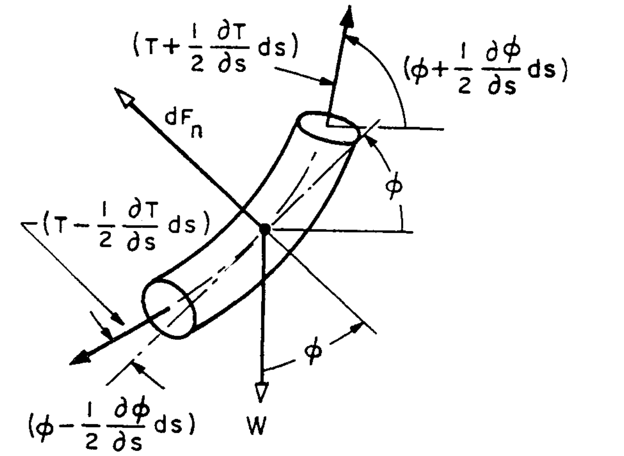

Fig. 3 Differential element of cable showing forces and angles.

On substituting into Eq. (6), we obtained for the longitudinal strain component in the element

$$\epsilon_z = \frac{1}{E}\left[\frac{T}{A} + 2\nu\rho g(h-z)\right] \tag{8}$$

It is seen immediately that the extension per unit length of the cable is not merely proportional to the tension but also contains a term proportional to the hydrostatic pressure. Thus, the extension of the cable is caused by two actions; pulling on the cable due to tension and squeezing of the cable due to hydrostatic pressure.

At this point, Poisson's ratio will be taken to be equal to ½ for all materials to be considered. Although it is true that Poisson's ratio is less than ½ for ferrous materials, it is also true that the theory of elasticity, which is being called upon, assumes the material to be isotropic, which it surely is not for the chains and cables usually employed in marine applications. The great simplifications introduced by assuming Poisson's ratio to be equal to ½ are felt to be justified on this basis. The reason this value of Poisson's ratio leads to simplifications is given in Ref. 2 (p. 10), where it is shown that for $\nu=½$ the volume expansion of an element is zero. Thus, the volume of a cylindrical element of length ds and cross-sectional area A_0 before stretching remains constant for this value of Poisson's ratio, so that, if the length of the element becomes $(1+\epsilon)ds$ and the cross-sectional area becomes A after stretching, it is clear that since volume is conserved

$$A = A_0/(1+\epsilon) \tag{9}$$

Then Eq. (5) becomes

$$dF_n = \rho g A_0(\cos\phi + (h-z/R))ds \tag{10}$$

To determine the first-order component of tension along the normal, we note that over the distance $(1+\epsilon)ds$ the tangent turns through an angular change of $d\phi$, and, hence, the contribution of the tension to the forces acting along the normal is

$$\frac{Td\phi}{(1+\epsilon)ds}\cdot(1+\epsilon)ds = \frac{Td\phi}{ds}$$

If w is the weight of the cable per unit length *in vacuo* of the unstretched cable, then the component along the normal of the unstretched cable is $-wds\cos\phi$. We arrive at the same form for the stretched cable since volume is conserved.

Finally, we consider that the current will induce a normal drag force per unit length on the cable of the form

$$\frac{dF}{ds} = \frac{1}{2}\rho V_c^2 (2a) C_N \tag{11}$$

Since volume is conserved during elongation of the cable, the radius a will become after stretching

$$a/(1+\epsilon)^{1/2}$$

Hence upon replacing ds by $(1+\epsilon)ds$ the current-induced force after stretching becomes for small ϵ

$$dF = \tfrac{1}{2}\rho V_c^2 (2a) C_N ds (1+\tfrac{1}{2}\epsilon) \tag{12}$$

The sum of all these static forces along the normal is for arbitrary ds

$$\frac{Td\phi}{ds} + \rho g A_0 (\cos\phi + \frac{h-z}{R}) - w\cos\phi - F(1+\frac{1}{2}\epsilon) = 0$$

Now, noting that

$$\frac{1}{R} = \frac{1}{(1+\epsilon)}\frac{d\phi}{ds} \tag{13}$$

We can combine terms to obtain

$$(T + \frac{\rho g A_0 (h-z)}{(1+\epsilon)})\frac{d\phi}{ds} - (w - \rho g A_0)\cos\phi - F(1+\frac{1}{2}\epsilon) = 0 \tag{14}$$

Since all the fluid pressures act along the normal, the balance of tangential forces requires that

$$\left[\frac{dT}{ds} - w\sin\phi + G(1+\frac{1}{2}\epsilon)\right] ds = 0 \tag{15}$$

where G, the tangential drag force per unit length induced by the current, is of the same form as the normal drag force F. The weight of the unstretched element in water is

$$w' = w - \rho g A_0 \tag{16}$$

and hence we see that w appears in the tangential equation whereas w' appears in the normal equation together with a term that modifies or alters the tension.

Let us define a quantity that we may call the effective tension T_e given by

$$T_e = T + \frac{\rho g A_0 (h-z)}{(1+\epsilon)} \tag{17}$$

Now differentiate with respect to s

$$\frac{dT_e}{ds} = \frac{dT}{ds} - \frac{\rho g A_0}{1+\epsilon}\frac{dz}{ds} + \rho g (h-z)\frac{d}{ds}\left(\frac{A}{1+\epsilon}\right) \tag{18}$$

But in the stretched condition

$$\frac{dz}{ds} = (1+\epsilon)\sin\phi \tag{19}$$

so that

$$\frac{dT}{ds} = \frac{dT_e}{ds} + \rho g A_0 \sin\phi - \rho g (h-z)\frac{d}{ds}\left(\frac{A_0}{1+\epsilon}\right) \tag{20}$$

Upon substituting Eq. (20) into Eqs. (15) and (16) and Eq. (17) into Eq. (14) we obtain

$$T_e \frac{d\phi}{ds} = w'\cos\phi + F(1+\frac{1}{2}\epsilon) \tag{21}$$

$$\frac{dT_e}{ds} = w'\sin\phi - G(1+\frac{1}{2}\epsilon) \tag{22}$$

where the term proportional to $(d/ds)\ [A_0/(1+\epsilon)]$ that appears in Eq. (20) has been dropped, since it is canceled identically by a longitudinal hydrostatic pressure force acting on the annulus between the section of area $A/(1+\epsilon)$ and the section of area $A/(1+\epsilon) + d[A/(1+\epsilon)]$.

In the case where current is absent $(F=G=0)$, it is seen immediately that [Eqs. (21) and (22) are identical to the usual form of the static cable equations with the cable weight in the liquid appearing in both equations. The departure from all previously published results is that the tension itself does not appear but only the effective tension. When current is present, the accounting for extensibility in the factor $(1+\tfrac{1}{2}\epsilon)$ that appears in both equations is different from any previously published result.

Now return to the expression for ϵ, Eq. (8). Upon setting $\nu = \tfrac{1}{2}$ and replacing A by $A_0/(1+\epsilon)$, this becomes

$$\epsilon = \frac{1}{E}\left[\frac{T(1+\epsilon)}{A_0} + \rho g (h-z)\right] \tag{23}$$

Eliminating T in favor of T_e by virtue of Eq. (17), we obtain for small ϵ

$$\epsilon = T_e / AE \tag{24}$$

This is the appropriate constitutive equation to account for hydrostatic pressure and shows that it is the effective tension and not the actual tension that controls the extensibility of the cable.

Finally, the tension itself may be determined by solving Eqs. (19), (21), and (22) subject to Eq. (24) and specified end conditions. The solution will yield $T_e(s)$, $\epsilon(s)$, and $z(s)$, which must then be substituted into Eq. (17) to yield $T(s)$. Note from Eq. (7) that the actual tension, not the effective tension, determines the longitudinal stress. Thus, the actual tension is critical when considering the ultimate strength of the cable.

Analysis of Unsteady Free Cables

To arrive at the net or resultant forces on the cable element when it is undergoing an arbitrary motion, we must join to the hydrostatic forces the forces generated by acceleration (added mass force plus a dynamic coupling due to the presence of the current, both of which are derived in Ref. 1). The resultant force must then be equated to the product of the mass and total acceleration of the element. In this way, the equations of motion of the cable are achieved. Defining U, V to be the tangential and normal velocities, respectively, and μ the mass per unit length, the normal and tangential inertial forces are, respectively

$$\mu(\frac{\partial V}{\partial t} + U\frac{\partial \phi}{\partial t})$$

$$\mu(\frac{\partial U}{\partial t} - V\frac{\partial \phi}{\partial t})$$

The hydrodynamic normal force derived in Ref. 1 is

$$\rho A(\frac{\partial V}{\partial t} + V_c\cos\phi\frac{\partial \phi}{\partial t})$$

Upon inserting into Eqs. (14) and (15) by virtue of Eq. (19), we ultimately arrive at the following normal and tangential force balance equations

$$\mu_1 \frac{\partial V}{\partial t} + (\mu_0 U + \rho A_0 V_c \cos\phi)\frac{\partial \phi}{\partial t} = T_e\frac{\partial \phi}{\partial s} - w'\cos\phi - (1+\frac{1}{2}\epsilon)F \tag{25}$$

$$\mu_0\left(\frac{\partial U}{\partial t}-V\frac{\partial\phi}{\partial t}\right)=\frac{\partial T_e}{\partial s}-w'\sin\phi+\left(1+\frac{1}{2}\epsilon\right)G \quad (26)$$

where, in parallel with Eq. (9)

$$\mu_0=\mu(1+\epsilon) \quad (27)$$

where μ_1 is the mass plus added mass per unit length before stretching.

In addition to the kinetic relationships (25) and (16), it is shown in Ref. 1 that the following kinematic or compatibility relationships must also be satisfied

$$\frac{\partial U}{\partial s}-V\frac{\partial\phi}{\partial s}=\frac{\partial\epsilon}{\partial t} \quad (28)$$

$$\frac{\partial V}{\partial s}+U\frac{\partial\phi}{\partial s}=(1+\epsilon)\frac{\partial\phi}{\partial t} \quad (29)$$

Now consider the constitutive equation for the cable. Classical elasticity has led us to Eq. (24), and it can be seen from this equation that a strain gauge on the cable is only capable of detecting the effective tension and not the actual tension. Reid derived a dynamical constitutive equation for a cable that takes into account hysteresis losses that may occur in dynamic situations.[3] This model is called a Maxwell model, and the constitutive equation for it is

$$\tau Y_0\frac{\partial\epsilon}{\partial t}+AE\epsilon=T+\tau\frac{\partial T}{\partial t} \quad (30)$$

where τ and Y_0 are specified constants.

For certain materials, the Maxwell model may represent a closer approximation to reality than the classical model. Unfortunately, the model does not take into account the effect of hydrostatic pressure, since this was not considered by Reid in his development. It is proposed that, in order to account for hydrostatic pressure in the Maxwell model, the property, derived on the basis of classical elasticity, that only the effective tension is detectable be retained, in which case the Maxwell model (including the effect of hydrostatic pressure) becomes

$$\tau Y_0\frac{\partial\epsilon}{\partial t}+AE\epsilon=T_e+\tau\frac{\partial T_e}{\partial t} \quad (31)$$

This reduces to Eq. (24) in the static case and may be used to replace it in the dynamic case.

Thus, the complete set of equations are (25), (26), (28), (29), (31), which constitute five equations for the five unknowns U, V, T_e, ϕ, ϵ.

Finally, it can be shown that the vertical and horizontal displacements, z, x are related to the normal and tangential velocities according to

$$\partial z/\partial t=V\cos\phi+U\cos\phi \quad (32)$$

$$\partial x/\partial t=U\cos\phi-V\cos\phi \quad (33)$$

Wave, Current, and Motion—Generated Hydrodynamic Forces on Cable Elements

There are several mechanisms that are responsible for the generation of hydrodynamic forces on cable elements. These may be classed as excitation forces, from the pressure and velocity fields arising from wave motion when the cable is considered to be held stationary and those arising from the relative motion of fluid and cable when the cable is oscillated in a horizontal current which, in general, varies with depth. We deal with each of these in turn.

Force From the Wave Pressure Field

This force is expected to be small, except possibly in shallow water, wherein the attenuation of dynamic effects with distance from the surface is much slower than in deep water.

Since the "scale" of the wave potential field is very large compared with the diameter of the cable, one may readily obtain an expression for the force arising from the dynamic pressure gradient by invoking the relation derived by Taylor[4]

$$F_j=-(1+k_j)V(\partial p/\partial x_j) \quad (34)$$

Here F_j is the force in the jth direction, k_j is the added mass coefficient of the body in the jth direction, V is the volume of the body, and p is the ambient, spatially varying pressure field (the gradient is evaluated at the center of volume). To apply Eq. (34) to any cable element immersed in the wave field we can write for the force density normal to the cable

$$F_n=-2\frac{A_0\rho}{(1+\epsilon)}(1+\epsilon)\,ds\frac{\partial^2\Phi}{\partial n\partial t}=-2A_0\rho\frac{\partial^2\Phi}{\partial n\partial t}ds \quad (35)$$

where we have taken $k_j=k_n=1$ for circular cross sections and related p to the potential function Φ for the wave motion by

$$p=\rho(\partial\Phi/\partial t) \quad (36)$$

For waves in two-dimensional space progressing in the negative x direction (from buoy to anchor) in water of finite depth h,

$$\Phi=\frac{a_w g}{\omega}\frac{\cosh kz}{\cosh kh}e^{i(kx+\omega t)} \quad (37)$$

where the wave number k is found from

$$k=(\omega^2/g)\tanh kh \quad (38)$$

The normal derivative in terms of cable angle ϕ is

$$\frac{\partial}{\partial n}=\sin\phi\frac{\partial}{\partial x}+\cos\phi\frac{\partial}{\partial z} \quad (39)$$

Carrying out the operations indicated in Eq. (35) by using Eq. (39) on Eq. (37) there is obtained after some manipulations, the following expression for the force per unit length due to the dynamic pressure gradient

$$F_p=\frac{2\rho gka_wA_0}{\cosh kh}\sin(\phi_0-ikz_0)e^{i(kx_0+\omega t)} \quad (40)$$

where k must be found from Eq. (38) for given values of ω and h and the subscript zero refers to the static equilibrium value.

Wave Velocity—Current Induced Viscous Forces

When wave motion is accompanied by a horizontal current V_c, there will be net normal and tangential forces associated with the convective pressure distribution around each cable element immersed in this disturbed, viscous fluid. Since the scale of the wave motion is large compared to the cable diameter, we may approximate the normal force acting on the cable (regarded as held stationary in the wave-current field) in a quasisteady sense by using the empirical "cross-flow drag" formula. This formula is based on the concept that the force normal to a circular cylinder inclined in a stream is proportional to the square of the velocity component normal to the cylinder. Here the generalization of Eq. (12) is taken to be

$$F_{total}=\tfrac{1}{2}\rho aC_R[V_c\sin\phi+V_w]^2(1+\tfrac{1}{2}\epsilon)ds \quad (41)$$

where V_w is the wave-induced velocity normal to the section, and C_R is the drag coefficient appropriate for the section determined for $\phi = \pi/2$, i.e., $C_N = C_R \sin^2\phi$. The deviation of this force from equilibrium values is taken to be

$$\delta F = \rho a C_R V_c V_w \sin\phi_0 \cdot (1 + \tfrac{1}{2}\epsilon_0 + \ldots)ds \qquad (42)$$

where it can be shown from Eq. (37) that

$$V_w = \frac{-ika_w g}{\omega \cosh kh} \sin(\phi_0 - ikz_0) e^{i(kx_0 + \omega t)} \qquad (43)$$

Thus, the excitation force per unit length arising from the coupling of wave and current velocities may be expressed as

$$\delta F = F_{w,c} = -i\left(1 + \frac{\epsilon_0}{2}\right) \frac{\rho g a_w k a C_R V_c \sin\phi_0}{\omega \cosh kh} \left\{ e^{i(kx_0 + \omega t)} \sin(\phi_0 - ikz_0) \right\} \qquad (44)$$

It is of interest to compare the modulus of F_p [Eq. (40] with that of $F_{w,c}$ [Eq. (44)] in order to determine whether either of these forces will dominate the other.

$$\frac{|F_{w,c}|}{|F_p|} = \frac{2(1 + \tfrac{1}{2}\epsilon_0) C_R V_c \sin\phi_0}{\pi \omega a} \qquad (45)$$

Let us neglect $\epsilon_0/2$ and estimate the practical values of the remaining parameters. The drag coefficient is of the order of unity (it can be effectively nearly 2.0 to account for cable roughness and strumming), V_c is of the order of 2 ft/sec, $\sin\phi_0 = 0(1)$, and ω is of the order of unity (1.25 for 5 sec waves) and ¼ in. $\leq a \leq 1$ in. If we take $a = 1/12$ ft, we see that $|F_{w,c}| / |F_p| \simeq 15$ for 1-in. cables and 30 for ½-in. cables. It would appear that unless the current speed $V_c = 0$ or it is very small the wave velocity-current induced normal force is an order of magnitude greater than that due to the force from the wave pressure gradient. Thus, the normal viscous force generally dominates the inertia force.

In a similar way, the tangential viscous force may be considered to be proportional to the square of the velocity component tangential to the cylinder, in which case the generalized tangential force including the wave-induced velocity is

$$G_{total} = \tfrac{1}{2}\rho a C_T [V_c \cos\phi + U_w]^2 (1 + \tfrac{1}{2}\epsilon) ds \qquad (46)$$

where U_w is the wave-induced velocity tangential to the section, C_T is the drag coefficient appropriate for the section determined for $\phi = 0$.

The deviation of this force from equilibrium values is

$$\delta G = \rho a C_T V_c U_w \cos\phi_0 (1 + \tfrac{1}{2}\epsilon_0 + \ldots) ds \qquad (47)$$

where, since the tangential derivative in terms of the cable angle is

$$\frac{\partial}{\partial s} = \cos\phi \frac{\partial}{\partial x} - \sin\phi \frac{\partial}{\partial z} \qquad (48)$$

it can be shown from Eq. (37) that

$$U_w = \frac{-ika_w g}{\omega \cosh kh} \cos(\phi_0 - ikz_0) e^{i(kx_0 + \omega t)} \qquad (49)$$

Thus the excitation force per unit length arising from coupling of wave and current velocities may be expressed as

$$\delta G = G_{w,c} = -\left(1 + \frac{\epsilon_0}{2}\right) \frac{\rho g a_w k a C_T V_c}{\omega \cosh kh} \cos(\phi_0 - ikz_0) e^{i(kx_0 + \omega t)} \qquad (50)$$

The ratio of the modulus of $G_{w,c}$ to $F_{w,c}$ is

$$\frac{|G_{w,c}|}{|F_{w,c}|} = \frac{C_T}{C_R} \operatorname{ctn}\phi_0 \qquad (51)$$

The ratio C_T/C_R is of the order 0.03. Hence, if we take $\operatorname{ctn}\phi_0$ to be 0(1), the ratio $|G_{w,c}| / |F_p| \simeq 0.5$ for 1-in. cables and 2 for ¼-in. cables for the same case that was used in the comparison of normal force. It would appear that unless the current speed $V_c = 0$ or it is very small the wave velocity-current induced tangential force is at most of the same order as that due to the force from the wave pressure gradient. It should be mentioned that in most practical cases $\operatorname{ctn}\phi_0$ will become large only near the sea bed where current speed is ordinarily very small, so the conclusion holds in this case also.

Normal and Tangential Forces Due to Cable Motions

The normal force per unit length due to the motion of the cable element along its local normal, in the presence of a current V_c, is taken to be given by

$$F = \tfrac{1}{2}\rho a C_R (V_c \sin\phi - V)^2 (1 + \tfrac{1}{2}\epsilon) \qquad (52)$$

The rate-of-change of this force with V is for $V_c >> V$

$$F_V = -\rho a C_R V_c \sin\phi_0 (1 + \tfrac{1}{2}\epsilon_0) \qquad (53)$$

We see that this force is independent of the tangential motion of the cable, i.e.

$$F_U = 0 \qquad (54)$$

As the cable changes angular position, we see that this force changes and hence

$$F_\phi = \rho a C_R V_c \cos\phi_0 (1 + \tfrac{1}{2}\epsilon_0) \qquad (55)$$

Now, in addition, we may account for the fact that the current velocity V_c varies with height above the sea bed as $V_c = V_c(z)$ and calculate

$$F_z = \rho a C_R V_c \sin\phi_0 (1 + \tfrac{1}{2}\epsilon_0)(dV_c/dz) \qquad (56)$$

It is expected that except in special circumstances this last force gradient will be small.

The total variation in normal force from cable motions can now be found from

$$F' = F_V V' + F_U U' + F_\phi \phi' + F_z z' \qquad (57)$$

with the coefficients given by Eq. (53) through (56). It is curious to note that the last two terms generally act like negative springs, $F_\phi \phi'$ always and $F_z z'$ whenever $dV_c/dz > 0$.

Similarly, the tangential force per unit length due to motion of the cable element along its tangent in the presence of a current is taken to be given by

$$G = \tfrac{1}{2}\rho a C_T (V_c \cos\phi - U)^2 (1 + \tfrac{1}{2}\epsilon) \qquad (58)$$

The rate of change of this force with V is for $V_c >> V$

$$G_U = -\rho a C_T V_C \sin\phi_0 (1 + \tfrac{1}{2}\epsilon_0) \qquad (59)$$

This force is independent of the normal motion of the cable, i.e.

$$G_V = 0 \qquad (60)$$

As the cable changes angular position, this force also changes, and hence

$$G_0 = \rho a C_T V_c \sin\phi_0 (1 + \tfrac{1}{2}\epsilon_0) \qquad (61)$$

Taking into account that the current velocity V_c varies with height above the sea bed, we find

$$G_z = \rho a C_T V_c \cos\phi_0 (1 + \tfrac{1}{2}\epsilon_0)(dV_c/dz) \qquad (62)$$

The total variation in tangential force from cable motions can now be found from

$$G' = G_V V' + G_U U' + G_\phi \phi' + G_z z' \qquad (63)$$

In both Eq. (57) and Eq. (63) the variable z' can be eliminated in favor of the velocities U' and V' by use of the kinematic relationship (32) which, in linearized form and in the frequency domain (ω), becomes

$$z' = (1/i\omega)\,[\,V'\cos\phi_0 + U'\sin\phi_0\,] \qquad (64)$$

Equations of Motion for Simple Harmonic Equations

We now consider a cable subjected to oscillatory wave forces along its length as well as motions imposed on one or both ends at the same frequency ω, so that the cable oscillates about an equilibrium configuration. We may separate out the time dependence by writing

$$U = U' e^{i\omega t} \qquad (65a)$$

$$V = V' e^{i\omega t} \qquad (65b)$$

$$\phi = \phi_0 + \phi' e^{i\omega t} \qquad (65c)$$

$$T_e = T_{e_0} + T_e' e^{i\omega t} \qquad (65d)$$

$$\epsilon = \epsilon_0 + \epsilon' e^{i\omega t} \qquad (65e)$$

where subscript zero indicates the equilibrium value and only the real part of the complex amplitudes and $e^{i\omega t}$ are to be retained. Here we limit ourselves to the case of an anchored cable, i.e., the equilibrium values of the cable velocity components $U_0\,V_0$ are zero.

Returning to Eqs. (25), (26), (28), (29), and (31) we obtain the following coupled ordinary differential equations upon retaining only the first-order perturbation terms

$$\frac{d}{ds}\begin{bmatrix} U' \\ V' \\ T_e' \\ \phi' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & a_{15} \\ a_{21} & 0 & 0 & a_{24} & 0 \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{53} & 0 & a_{55} \end{bmatrix} \begin{bmatrix} U' \\ V' \\ T_e' \\ \phi' \\ \epsilon' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_{w,c}' \\ (F_p' + F_{w,c}') \\ 0 \end{bmatrix} \qquad (66)$$

where

$$a_{12} = d\phi_0/ds; \quad a_{15} = i\omega \qquad (67a,b)$$

$$a_{21} = -d\phi_0/ds; \quad a_{24} = i\omega(1+\epsilon_0) \qquad (67c,d)$$

$$a_{31} = i\omega\mu - (1 + \frac{1}{2}\epsilon_0)\,(G_U + \frac{G_z \sin\phi_0}{i\omega}\,) \qquad (67e)$$

$$a_{32} = -\,(1 + \frac{1}{2}\epsilon_0)\,\frac{G_z \cos\phi_0}{i\omega} \qquad (67f)$$

$$a_{34} = w'\cos\phi_0 - (1 + ½\epsilon_0)G_\phi;$$

$$a_{35} = -G_0/2 \qquad (67g,h)$$

$$a_{41} = \frac{(1+½\epsilon_0)\cos\phi_0 F_z}{i\omega T_{e_0}} \qquad (67i)$$

$$a_{42} = \left[\,i\omega\mu_t + (1+\frac{1}{2}\epsilon_0)\,(F_V + \frac{F_z}{i\omega}\cos\phi_0\right]\frac{1}{T_{e_0}} \qquad (67j)$$

$$a_{43} = -\frac{d\phi_0}{ds}\,\frac{1}{T_{e_0}} \qquad (67k)$$

$$a_{44} = \frac{1}{T_{e_0}}\,(i\omega\rho A_0 V_c \cos\phi_0 - w'\sin\phi_0 + (1 + \frac{1}{2}\epsilon_0)F_\phi) \qquad (67l)$$

$$a_{45} = F_0/2T_{e_0} \qquad (67m)$$

$$a_{53} = (1 + i\omega\tau); \quad a_{55} = -\,(i\omega\tau Y_0 + AE) \qquad (67n,o)$$

In deducing these results products of $\epsilon'/2$ and the exciting force amplitudes F_p', $F_{w,c}'$ and $G_{w,c}'$ have been neglected. These latter quantities are the coefficients of $e^{i\omega t}$ in Eqs. (40), (44), and (49), respectively, with the coordinates x, z replaced by their equilibrium values.

To solve the set of equations by a numerical procedure it is, of course, necessary to find the equilibrium solution for the cable geometry, effective tension, and strain. These results enable the evaluation of the matrix elements (67). The dynamic problem is then well-posed once the conditions on the cable ends are specified.

Boundary Conditions

Equation (66) constitutes a set of four simultaneous differential equations for the four state variables U', V', T_e', ϕ'. Four boundary conditions are therefore required in order to specify the problem. It is not difficult to show that these break up into two conditions at the buoy attachment point and two at the cable touchdown point. The two conditions at the buoy attachment point are nonhomogenous since the buoy itself is being forced by the waves. The manner in which these wave forces are transmitted to the cable will not be considered here in detail however, since the complexities of the buoy motions themselves would have to be taken into account, and this is beyond the scope of the present work. The two conditions at the cable touchdown point will now be developed. It is easy to see that if the touchdown point conicides with the anchor the cable must be stationary there and so, under these circumstances, the conditions are

$$U = V = 0 \quad \text{at} \quad s = 0 \qquad (68)$$

When the cable lies along the sea bed between the touchdown point and the anchor, on the other hand, the cable, which is elastic, is capable of transmitting longitudinal waves between the two points and, as a consequence, the tangential velocity U no longer vanishes there but instead obeys the equations

$$\partial U/\partial s = \partial\epsilon/\partial t$$

$$\mu(\partial U/\partial t) = \partial T_e/\partial s \qquad (69)$$

It can be inferred from these equations that along the sea bed the steady-state component of the effective tension is constant. Attention will be confined to sinusoidal dynamic motions. Then since $\epsilon = KT_e$ where $K = -a_{53}/a_{55}$ these partial differential equations reduce to the following pair of ordinary differential equations

$$dU'/ds = i\omega K T_e' \tag{70a}$$

$$dT'/ds = \mu i\omega U' \tag{70b}$$

the general solutions to which are

$$U' = a\cos\alpha s + B\sin\alpha s \tag{71a}$$

$$T_e' = \frac{\alpha i}{\omega K}[A\cos\alpha s - B\cos\alpha s] \tag{71b}$$

where $\alpha = \omega(\mu k)^{1/2}$ and A and B are constants of integration. Assume the cable along the sea bed to be uniform. The touchdown point lies at $s=0$ while the anchor lies at $s=-L$. Since the anchor is permanently fixed, the condition there is

$$U' = 0 \quad \text{at } s=-L \tag{72}$$

Upon substituting into Eq. (71), a relationship is obtained between A and B from which A may be eliminated. Finally, eliminating B between the resulting equations we find the following relationship between U' and T_e' all along the cable between the anchor and the touchdown point

$$U' + iT_e'\frac{\omega K}{\alpha}\left[\frac{\tan\alpha L\cos\alpha s + \sin\alpha s}{\tan\alpha L\sin\alpha s - \cos\alpha s}\right] = 0$$

Upon setting $s=0$ we find the boundary condition that is to be applied at the touchdown point:

$$U' - \beta T_e' = 0 \quad \text{at} \quad s=0 \tag{73}$$

where

$$\beta = (K/\mu)^{1/2}\tan\alpha L \tag{74}$$

It is to be noted that if either $L=0$ (touchdown point at the anchor) or $K-0$ (inelastic cable) the boundary condition reduces, as it must, to $U'=0$. The second boundary condition is still, of course, $V'=0$ at $s=0$.

Equations (73) and (74 can be generalized to account for the fact that the cable lying along the sea bed is not uniform but instead consists of several uniform segments. In this case, the constants K and μ will change from segment to segment, and the general solution (71) will be valid in each segment but with different values for A and B. Thus, in the jth segment we have

$$U_j' = A_j\cos\alpha_j s + B_j\sin\alpha_j s \tag{75a}$$

$$T_{e_j}' = \frac{\alpha_j i}{\omega K_j}[A_j\sin\alpha_j s - B_j\cos\alpha_j s] \tag{75b}$$

The effective tension and velocity must be continuous at the joint between two contiguous segments, and these conditions can be used to obtain recursion formulas for the A_j's and B_j's. Thus, upon defining s to be zero at the right-hand end of each segment and to be equal to $-L_j$ at the left-hand end of the jth segment, we obtain the following recursion formulas

$$A_j = A_{j+1}\cos(\alpha_{j+1}L_{j+1}) - B_{j+1}\sin(\alpha_{j+1}L_{j+1}) \tag{76a}$$

$$\left(\frac{K_{j+1}\alpha_j}{K_j\alpha_{j+1}}\right)B_j = A_{j+1}\sin(\alpha_{j+1}L_{j+1}) + B_{j+1}\cos(\alpha_{j+1}L_{j+1}) \tag{76b}$$

If these equations are solved for A_{j+1} and B_{j+1} and the ratio taken, the result is

$$C_{j+1} \equiv \frac{A_{j+1}}{B_{j+1}} = \frac{\left(\frac{K_{j+1}\alpha_j}{K_j\alpha_{j+1}}\right)\sin(\alpha_{j+1}L_{j+1}) + C_j\cos(\alpha_{j+1}L_{j+1})}{\left(\frac{K_{j+1}\alpha_j}{K_j\alpha_{j+1}}\right)\cos(\alpha_{j+1}L_{j+1}) - C_j\sin(\alpha_{j+1}L_{j+1})} \tag{77}$$

At the anchor, $U'=0$, which leads to the following starting condition for the recursion

$$C_1 = \tan\alpha_1 L_1 \tag{78}$$

Finally, if there are N segments, then, upon setting $j=N$ and taking the ratio of U_N' to T_{e_N}' at $s=0$, the boundary condition is seen to be (73) but with β given by

$$\beta = \frac{\omega K_N}{\alpha_N}C_N \tag{79}$$

Finally, it can be shown that the two conditions at the buoy are of the form

$$U' = U_w + U_T T_e' + U_\phi \phi' \tag{80}$$

$$\text{at } s=\ell$$

$$V' = V_w + V_T T_e' + V_\phi \phi' \tag{81}$$

where U_w and V_w are tangential and normal wave velocities, respectively, at the buoy and the quantities U_T, U_ϕ, V_T, V_ϕ are frequency-dependent complex parameters that depend on the buoy dynamic characteristics and the location of the cable attachment point.

Solving the Dynamic Equations

The dynamic equation (66) can be written

$$dx/ds = Ax + F \tag{82}$$

where

$$x = \begin{vmatrix} U' \\ V' \\ T_e' \\ \phi' \end{vmatrix} \tag{83}$$

and the algebraic equation for ϵ' is eliminated by substitution. The boundary conditions are

$$U' - i\beta T_e' = 0, \quad V' = 0 \quad \text{at } s=0 \tag{84}$$

together with Eq. (80) and (81). One method for solving this two-point boundary value problem is as follows:

Let

$$x_I(s) = \begin{bmatrix} x_{1_I}(s) \\ x_{2_I}(s) \\ x_{3_I}(s) \\ x_{4_I}(s) \end{bmatrix}, x_{II} = \begin{bmatrix} x_{1_{II}}(s) \\ x_{2_{II}}(s) \\ x_{3_{II}}(s) \\ x_{4_{II}}(s) \end{bmatrix} \tag{85}$$

denote the solutions of the homogeneous counterpart of Eq. (82), i.e., with $F=0$, and with initial conditions

$$x_I(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad x_{II}(0) = \begin{bmatrix} 0 \\ i\beta \\ 1 \\ 0 \end{bmatrix} \tag{86}$$

and let

$$x_{III} = \begin{bmatrix} x_{1_{III}}(s) \\ x_{2_{III}}(s) \\ x_{3_{III}}(s) \\ x_{4_{III}}(s) \end{bmatrix} \quad (87)$$

denote a solution of the complete equation (82) with the homogeneous initial conditions

$$x_{III}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (88)$$

Then the solution to Eq. (82) with boundary conditions (80), (81), and (84) is given by

$$x(s) = a_I x_I(s) + a_2 x_{II}(s) + x_{III}(s) \quad (89)$$

where the constants a_I and a_2 as well as $\phi(\ell)$ and $T_e'(\ell)$ are obtained by solving the following system of equations

$$\begin{bmatrix} x_{I_I}(\ell) & x_{I_{II}}(\ell) & -V_\phi & -V_T \\ x_{2_I}(\ell) & x_{2_{II}}(\ell) & -U_\phi & -U_T \\ x_{3_I}(\ell) & x_{3_{II}}(\ell) & 0 & -1 \\ x_{4_I}(\ell) & x_{4_{II}}(\ell) & -1 & 0 \end{bmatrix} \begin{bmatrix} a_I \\ a_2 \\ \phi'(\ell) \\ T_e'(\ell) \end{bmatrix} = \begin{bmatrix} V_w \\ U_w \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x_{I_{III}}(\ell) \\ x_{2_{III}}(\ell) \\ x_{3_{III}}(\ell) \\ x_{4_{III}}(\ell) \end{bmatrix} \quad (90)$$

The two-point boundary value problem is then solved by calculating x_I, x_{II}, and x_{III}, determining a_I and a_2 from (90), and substituting into (89). The vectors x_I, x_{II}, and x_{III} are linearly independent, that is, no complex numbers a and b exist such that $ax_I + bx_{II} + x_{III} = 0$ for any s. However, under certain conditions of cable length and frequency this may not be so to within the accuracy of computers. In these cases, the matrix in Eq. (90) becomes ill-conditioned and a_I and a_2 cannot be determined. To preclude this phenomenon, the vectors x_I, x_{II}, and x_{III} must be renormalized in stages according to a method of Godonov.[5] The details cannot be presented here for lack of space. However, it should be mentioned that the method is developed by Godonov for equations given in terms of real variables. In using the method, the scalar product of two vectors is required and this has been generalized to complex variables by Trampus[6] as follows

If

$$x = \begin{bmatrix} x_I \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \; y = \begin{bmatrix} y_I \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

then

$$x \cdot y = \bar{x}_I y_I + \bar{x}_2 y_2 + \bar{x}_3 y_3 + \bar{x}_4 y_4 \quad (91)$$

where $\bar{x}_i$ is the complex conjugate of x_i, $i = 1, 2, 3, 4$.

Conclusions

The statics and dynamics of anchoring cables have been analyzed, and, in the course of doing so, several new aspects of this time-honored problem have been discovered. In particular, the effects of hydrostatic pressure is accounted for; the strain of the cable is accounted for properly; the effect of cable angle on the added mass of a cable in a stream is included; wave forces acting on the cable are analyzed; the effect of waves on the dynamic boundary condition at touchdown of slack cables is determined; the complete effect of current, including current gradient, on the cable forces is found. Finally, a method is outlined for finding the numerical solution of the linearized cable-buoy dynamic equations subject to appropriate boundary conditions that overcomes the tendency of solutions to become linearly dependent, a problem that has previously plagued all numerical analyses of long cables in the frequency domain.

Unfortunately, space does not permit the presentation of the numerical results of a computer program that has been developed based on the analysis presented here. Such a program does indeed exist and resides at the NOAA Data Buoy Office.

References

[1] Breslin, J.P., "Dynamic Forces Exerted by Oscillating Cables," *Journal of Hydronautics*, Vol. 8, Jan. 1974, pp. 19-31.

[2] Timoshenko, S., *Theory of Elasticity*, McGraw-Hill, New York, 1934.

[3] Reid, R.O., "Dynamics of Deep-Sea Mooring Lines," Texas A&M University Project 204, Reference 68-11F, July 1968.

[4] Taylor, G.I., "The Forces on a Body Placed in a Curved or Converging Stream of Fluid," *Proceedings of the Royal Society*, Series A: *Mathematical and Physical Sciences*, Vol. CXX, 1928, pp. 260–283.

[5] Godonov, S.K., "On the Numerical Solutions of Boundary Problems for Systems of Linear Ordinary Differential Equations," translated from *Uspekhi Mathematischeskikh Nauk*, Vol. 16, May-June 1961, pp. 171–174; Department of Commerce Report 63-10299, 1963.

[6] Trampus, A., private communication, 1972.